Random Regressions

LRS

CGIL

July-Aug 2012

Where Needed

- Test day milk yields of cows, goats, sheep
- Daily feed intake of pigs, cattle, goats
- Daily weight or height changes
- Blood components per minute or hour after treatment (diet, injections)
- Any trait measured over time (longitudinal)
- Measurements that follow a trajectory

Pig Weights

Pigs were measured on days 10, 20, 30, 40, 50, and 60 while on growth test. The covariance matrix of weights was

$$\mathbf{V} = \begin{pmatrix} 2.5 & 4.9 & 4.6 & 4.6 & 4.3 & 4.0 \\ 4.9 & 13.5 & 12.1 & 12.3 & 11.9 & 10.7 \\ 4.6 & 12.1 & 15.2 & 14.5 & 14.6 & 12.5 \\ 4.6 & 12.3 & 14.5 & 20.0 & 19.0 & 16.9 \\ 4.3 & 11.9 & 14.6 & 19.0 & 25.0 & 20.3 \\ 4.0 & 10.7 & 12.5 & 16.9 & 20.3 & 30.0 \end{pmatrix}$$

Covariance Functions

Kirkpatrick et al. (1991) wanted to estimate variances and covariances for other days during the interval 10 to 60 d.

$$\mathbf{V} = \Phi \mathbf{H} \Phi'$$

where Φ was based on Legendre polynomials and time.

Legendre polynomials, 1797

$$\begin{aligned} P_0(x) &= 1, \quad \text{and} \\ P_1(x) &= x, \end{aligned}$$

then, in general, the $n+1$ polynomial is described by the following recursive equation:

$$P_{n+1}(x) = \frac{1}{n+1}\left((2n+1)xP_n(x) - nP_{n-1}(x)\right)$$

These quantities are "normalized" using

$$\phi_n(x) = \left(\frac{2n+1}{2}\right)^{.5} P_n(x).$$

LP

This gives the following series,

$$
\begin{aligned}
\phi_0(x) &= \left(\frac{1}{2}\right)^{.5} P_0(x) = .7071 \\
\phi_1(x) &= \left(\frac{3}{2}\right)^{.5} P_1(x) \\
&= 1.2247x \\
P_2(x) &= \frac{1}{2}(3xP_1(x) - 1P_0(x)) \\
\phi_2(x) &= \left(\frac{5}{2}\right)^{.5} (\frac{3}{2}x^2 - \frac{1}{2}) \\
&= -.7906 + 2.3717x^2
\end{aligned}
$$

Legendre polynomials

$$\Lambda' = \begin{pmatrix} .7071 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.2247 & 0 & 0 & 0 & 0 \\ -.7906 & 0 & 2.3717 & 0 & 0 & 0 \\ 0 & -2.8062 & 0 & 4.6771 & 0 & 0 \\ .7955 & 0 & -7.9550 & 0 & 9.2808 & 0 \\ 0 & 4.3973 & 0 & -20.5206 & 0 & 18.4685 \end{pmatrix}$$

$$q_\ell = -1 + 2\left(\frac{t_\ell - t_{min}}{t_{max} - t_{min}}\right)$$

$$t_{min} = 10, \quad t_{max} = 60$$

Days

Days on Test	Age	Standardized Value
10	31	-1.000
20	41	-.600
30	51	-.200
40	61	.200
50	71	.600
60	81	1.000

Days

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -.600 & .360 & -.216 & .130 & -.078 \\ 1 & -.200 & .040 & -.008 & .002 & -.000 \\ 1 & .200 & .040 & .008 & .002 & .000 \\ 1 & .600 & .360 & .216 & .130 & .078 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Φ

$$
\begin{aligned}
\Phi &= \mathbf{M}\Lambda, \\
&= \begin{pmatrix}
.7071 & -1.2247 & 1.5811 & -1.8708 & 2.1213 & -2.3452 \\
.7071 & -.7348 & .0632 & .6735 & -.8655 & .3580 \\
.7071 & -.2449 & -.6957 & .5238 & .4921 & -.7212 \\
.7071 & .2449 & -.6957 & -.5238 & .4921 & .7212 \\
.7071 & .7348 & .0632 & -.6735 & -.8655 & -.3580 \\
.7071 & 1.2247 & 1.5811 & 1.8708 & 2.1213 & 2.3452
\end{pmatrix}
\end{aligned}
$$

Getting **H**

$$\begin{aligned}
\mathbf{V} &= \Phi\mathbf{H}\Phi' \\
\mathbf{H} &= \Phi^{-1}\mathbf{V}\Phi^{-T}, \\
&= \begin{pmatrix}
27.69 & 5.29 & -1.95 & 0.05 & -1.17 & 0.52 \\
5.29 & 4.99 & 0.42 & -0.25 & -0.30 & -0.75 \\
-1.95 & 0.42 & 1.51 & 0.20 & -0.33 & -0.07 \\
0.05 & -0.25 & 0.20 & 1.19 & 0.06 & -0.71 \\
-1.17 & -0.30 & -0.33 & 0.06 & 0.58 & 0.15 \\
0.52 & -0.75 & -0.07 & -0.71 & 0.15 & 1.12
\end{pmatrix}
\end{aligned}$$

Days 25 and 55 on Test

Standardized time is -0.4 and 0.8, respectively.

$$\mathbf{L} = \left(\begin{array}{rrrrrr} .7071 & -.4899 & -.4111 & .8232 & -.2397 & -.6347 \\ .7071 & .9798 & .7273 & .1497 & -.4943 & -.9370 \end{array} \right)$$

$$\mathbf{LHL}' = \left(\begin{array}{rr} 14.4226 & 13.7370 \\ 13.7370 & 28.9395 \end{array} \right)$$

Pig Weights

Pigs were measured on days 10, 20, 30, 40, 50, and 60 while on growth test. The covariance matrix of weights was

$$\mathbf{V} = \left(\begin{array}{cccccccc} 2.5 & 4.9 & & 4.6 & 4.6 & 4.3 & & 4.0 \\ 4.9 & 13.5 & & 12.1 & 12.3 & 11.9 & & 10.7 \\ & & 14.42 & & & & 13.74 & \\ 4.6 & 12.1 & & 15.2 & 14.5 & 14.6 & & 12.5 \\ 4.6 & 12.3 & & 14.5 & 20.0 & 19.0 & & 16.9 \\ 4.3 & 11.9 & & 14.6 & 19.0 & 25.0 & & 20.3 \\ & & 13.74 & & & & 28.94 & \\ 4.0 & 10.7 & & 12.5 & 16.9 & 20.3 & & 30.0 \end{array} \right)$$

Lower Orders

- Order 6 fit **V** perfectly.
- Is there a lower order fit that works almost as well?
- Order 3 would use only the first 3 columns of Φ. $\mathbf{H}^{\star}$ has dimension 3×3.

$$\Phi^{*} = \begin{pmatrix} .7071 & -1.2247 & 1.5811 \\ .7071 & -.7348 & .0632 \\ .7071 & -.2449 & -.6957 \\ .7071 & .2449 & -.6957 \\ .7071 & .7348 & .0632 \\ .7071 & 1.2247 & 1.5811 \end{pmatrix},$$

Order 3

$$
\begin{aligned}
\Phi^{*'}\mathbf{V}\Phi^{*} &= \Phi^{*'}(\Phi^{*}\mathbf{H}^{*}\Phi^{*'})\Phi^{*} \\
&= (\Phi^{*'}\Phi^{*})\mathbf{H}^{*}(\Phi^{*'}\Phi^{*}) \\
\mathbf{H}^{*} &= (\Phi^{*'}\Phi^{*})^{-1}\Phi^{*'}\mathbf{V}\Phi^{*}(\Phi^{*'}\Phi^{*})^{-1}
\end{aligned}
$$

Order 3

$$
\begin{aligned}
(\Phi^{*'}\Phi^{*})^{-1} &= \begin{pmatrix} .3705 & .0000 & -.0832 \\ .0000 & .2381 & .0000 \\ -.0832 & .0000 & .1860 \end{pmatrix} \\
\Phi^{*'}\mathbf{V}\Phi^{*} &= \begin{pmatrix} 220.2958 & 78.0080 & 61.4449 \\ 78.0080 & 67.5670 & 44.9707 \\ 61.4449 & 44.9707 & 50.5819 \end{pmatrix} \\
\mathbf{H}^{*} &= \begin{pmatrix} 26.8082 & 5.9919 & -2.9122 \\ 5.9919 & 3.8309 & .4468 \\ -2.9122 & .4468 & 1.3730 \end{pmatrix}
\end{aligned}
$$

Testing

$$\mathbf{V}^* = \Phi^* \mathbf{H}^* \Phi^{*'}$$

Let

$$\mathbf{E} = \mathbf{V}^* - \mathbf{V}$$

Square the diagonals and elements above the diagonal and sum together, the value is 59.3476, it has 6 df. Do the same for order 5, giving 7.2139

with 15 df.

$$\sigma^2 = 7.2139/6 = 1.2023.$$

$$\begin{aligned} F &= \frac{(\hat{\mathbf{e}}'\hat{\mathbf{e}}_{m=3} - \hat{\mathbf{e}}'\hat{\mathbf{e}}_{m=5})/(15-6)}{\sigma^2} \\ &= \frac{(59.3476 - 7.2139)/9}{1.2023} \\ &= 5.7926/1.2023 = 4.8180 \end{aligned}$$

Random Regressions

- Presented by Henderson in his book, 1984.
- Earlier paper with his son.
- LRS presented application to dairy cows, 1994.
- Implemented in Canada, 1999.

Model Considerations

- Need to model curve shape of observations over time, using LP, other covariates, or intervals of time.
- Many fixed curves may be needed, by breed, year, age at calving, parity, etc.
- All factors in model have curves, the effects change with the age of the animal, or the number of days in milk.
- Residual variance likely changes over time too.
- PE effects needed, following LP.

Pig Test Station Model

Table 1
Traits and Factors.

Trait	Trial-Breed-Sex	Machine type	Technician	Trial-Pen	Animal Permanent Environ.	Animal Genetic
Visits	X			X	X	X
Duration	X			X	X	X
Feed Intake	X			X	X	X
Weight	X			X	X	X
Backfat	X	X	X	X	X	X
Lean	X	X	X	X	X	X

Example

- Assume two pigs, one at 25 days and one at 55 days.
- Both belong to same trial
- Assume LP of order 2,

$$\mathbf{X} = \begin{pmatrix} .7071 & -.4899 & -.4111 \\ .7071 & .9798 & .7273 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 1.09 \\ 2.97 \end{pmatrix}$$

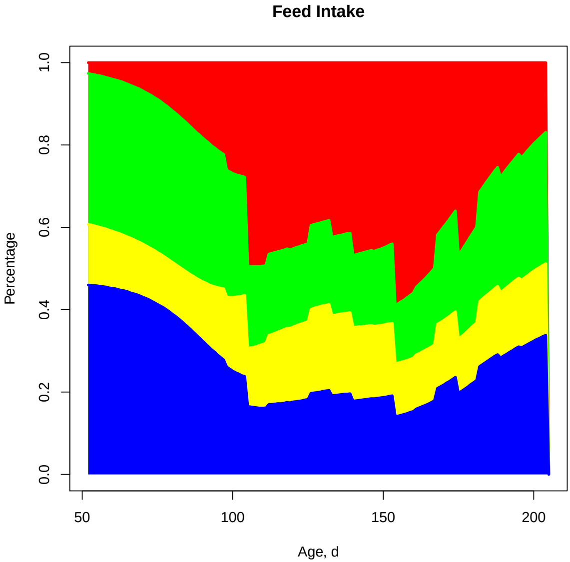
  for trial effects.
- For animal additive genetic and PE matrices

$$\mathbf{Z} = \left(\begin{array}{rrr|rrr} .7071 & -.4899 & -.4111 & 0 & 0 & 0 \\ 0 & 0 & 0 & .7071 & .9798 & .7273 \end{array} \right)$$

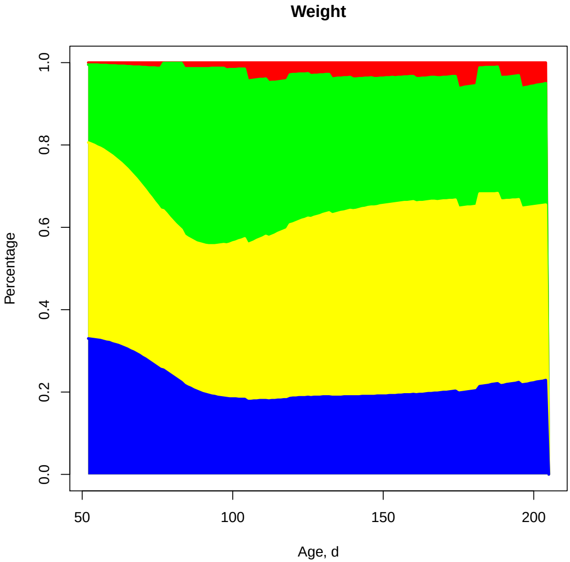
Meyer and Hill

- Showed that Covariance Functions and Random Regression Models were the same.
- RRM estimates $\mathbf{H}^*$ directly from data, and for genetic and PE effects, and any other random effects.
- See examples in Exercises, with R.

Percentage Variation

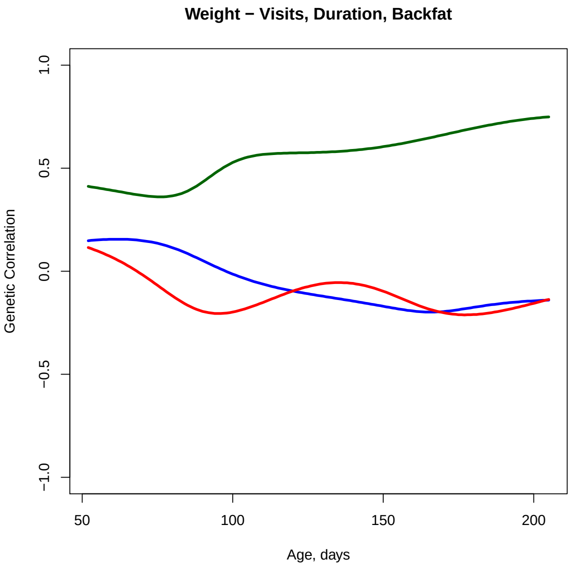


Percentage Variation

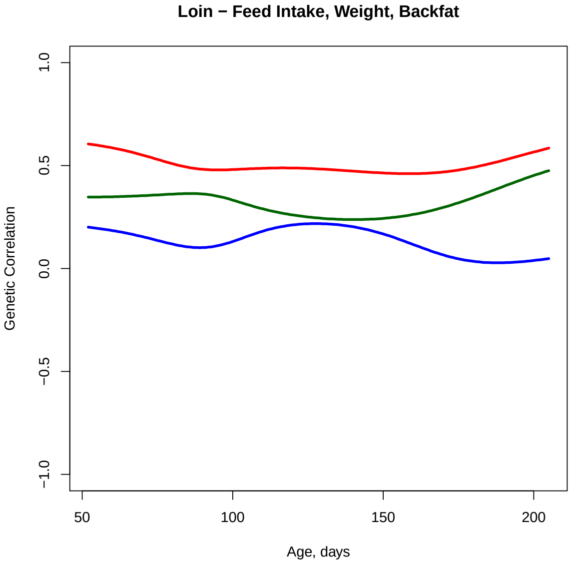


Percentage Variation

Correlations



Correlations



Correlations