Matrix Algebra

LRS

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Vectors

$$\mathbf{w} = \begin{pmatrix} 72 \\ 18 \\ 54 \\ 37 \end{pmatrix}$$

The transpose of **w** is

$$\mathbf{w}' = \begin{pmatrix} 72 & 18 & 54 & 37 \end{pmatrix}$$

Matrices

$$\mathbf{A} = \begin{pmatrix} 7 & 18 & -2 & 22 \\ -16 & 3 & 55 & 1 \\ 9 & -4 & 0 & 31 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} x & y+1 & x+y+z \\ a-b & c\log d & e \\ \sqrt{x-y} & (m+n)/n & p \end{pmatrix}$$

and

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix}$$

Transpose

$$\mathbf{A} = \begin{pmatrix} 7 & 18 & -2 & 22 \\ -16 & 3 & 55 & 1 \\ 9 & -4 & 0 & 31 \end{pmatrix}$$

and

$$\mathbf{A}' = \begin{pmatrix} 7 & -16 & 9 \\ 18 & 3 & -4 \\ -2 & 55 & 0 \\ 22 & 1 & 31 \end{pmatrix}$$

Matrix Addition

Conformable for Addition Rule

Two matrices are conformable for addition if they have the same number of rows and columns.

$$
\begin{aligned}
\mathbf{C} &= \mathbf{A} + \mathbf{B} \\
c_{ij} &= a_{ij} + b_{ij}
\end{aligned}
$$

Example

Let

$$\mathbf{A} = \begin{pmatrix} 2 & 9 \\ 7 & -3 \\ -5 & -4 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & -8 \\ -5 & 5 \\ 6 & 7 \end{pmatrix}$$

Both have 3 rows and 2 columns, then

$$\mathbf{C} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix}$$

Types of Matrices

- Rectangular, $r \neq c$
- Square, $r = c$

$$\mathbf{P} = \begin{pmatrix} 2 & 4 & -6 & 1 \\ 1 & 3 & -5 & 0 \\ 4 & 1 & 7 & -3 \\ -2 & -1 & -4 & 8 \end{pmatrix}$$

- Symmetric ($\mathbf{T} = \mathbf{T}'$)

$$\mathbf{T} = \begin{pmatrix} 2 & 4 & -6 & 1 \\ 4 & 3 & -5 & 0 \\ -6 & -5 & 7 & -3 \\ 1 & 0 & -3 & 8 \end{pmatrix}$$

More Types

- Diagonal

$$\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

- Identity
- Null
- Matrices With Only 1's
- Triangular, upper or lower

$$\mathbf{T} = \begin{pmatrix} 2 & 4 & -6 & 1 \\ 0 & 3 & -5 & 0 \\ 0 & 0 & 7 & -3 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

One More Type

- Tridiagonal

$$\mathbf{B} = \begin{pmatrix} 10 & 3 & 0 & 0 & 0 & 0 \\ 3 & 10 & 3 & 0 & 0 & 0 \\ 0 & 3 & 10 & 3 & 0 & 0 \\ 0 & 0 & 3 & 10 & 3 & 0 \\ 0 & 0 & 0 & 3 & 10 & 3 \\ 0 & 0 & 0 & 0 & 3 & 10 \end{pmatrix}$$

Matrix Multiplication

Conformable for Multiplication Rule

Two matrices are conformable for multiplication if the number of columns in the first matrix equals the number of rows in the second matrix.

$$\mathbf{C}_{p\times q} = \{c_{ij}\}$$

and

$$\mathbf{D}_{m\times n} = \{d_{ij}\}$$

and $q = m$, then

$$\mathbf{CD}_{p\times n} = \{\sum_{k=1}^{m} c_{ik} d_{kj}\}$$

Example

$$\mathbf{C} = \begin{pmatrix} 6 & 4 & -3 \\ 3 & 9 & -7 \\ 8 & 5 & -2 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{pmatrix}$$

$$\mathbf{CD} = \left(\begin{array}{c|c} 6(1)+4(2)-3(3) & 6(1)+4(0)-3(-1) \\ 3(1)+9(2)-7(3) & 3(1)+9(0)-7(-1) \\ 8(1)+5(2)-2(3) & 8(1)+5(0)-2(-1) \end{array}\right) = \begin{pmatrix} 5 & 9 \\ 0 & 10 \\ 12 & 10 \end{pmatrix}$$

Transpose of A Product

The transpose of the product of two or more matrices is the product of the transposes of each matrix in reverse order.

$$(\mathbf{CDE})' = \mathbf{E}'\mathbf{D}'\mathbf{C}'$$

Special Products

- Idempotent

$$\mathbf{AA} = \mathbf{A}$$

- Nilpotent

$$\mathbf{BB} = \mathbf{0}$$

- Orthogonal

$$\mathbf{UU}' = \mathbf{I}$$

which also implies that

$$\mathbf{U}'\mathbf{U} = \mathbf{I}$$

provided that $\mathbf{U}$ is square.

Some Rules on Multiplication

1. Multiplication of a matrix by a scalar results in multiplying every element of the matrix by that scalar.
2. $\mathbf{ABC} = \mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.
3. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.
4. $(\mathbf{A} + \mathbf{B})^2 = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{AA} + \mathbf{AB} + \mathbf{BA} + \mathbf{BB}$.

Traces

Traces of square matrices only.

Let

$$\mathbf{T} = \begin{pmatrix} 2 & 4 & -6 & 1 \\ 4 & 3 & -5 & 0 \\ -6 & -5 & 7 & -3 \\ 1 & 0 & -3 & 8 \end{pmatrix}$$

then

$$tr(\mathbf{T}) = 2 + 3 + 7 + 8 = 20$$

Traces are associated with degrees of freedom in hypothesis testing.

Traces

Rotation Rule

$$tr(\mathbf{ABC}) = tr(\mathbf{BCA}) = tr(\mathbf{CAB})$$

Determinants

Determinants exist only for square matrices.

Let

$$\mathbf{A} = \begin{pmatrix} 6 & 2 \\ 1 & 4 \end{pmatrix}$$

$$\mid \mathbf{A} \mid = (6)(4) - (1)(2) = 22$$

General Expression

$$\mid \mathbf{A} \mid = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \mid \mathbf{M}_{ij} \mid$$

where $\mathbf{M}_{ij}$ is the minor of element a_{ij} obtained by deleting the i^{th} row and j^{th} column of $\mathbf{A}$.

Example Determinant, 3×3 Matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 2 & 7 \\ 6 & 1 & 8 \\ 4 & 3 & 9 \end{pmatrix}$$

$$|\,\mathbf{A}\,| = 5\begin{vmatrix} 1 & 8 \\ 3 & 9 \end{vmatrix} - 2\begin{vmatrix} 6 & 8 \\ 4 & 9 \end{vmatrix} + 7\begin{vmatrix} 6 & 1 \\ 4 & 3 \end{vmatrix}$$

$$|\,\mathbf{A}\,| = 5(-15) - 2(22) + 7(14) = -21$$

Any row or column of **A** can be used, and the same value of the determinant will be obtained.

Matrix Inversion

Inverse of a square matrix, $\mathbf{A}$, with a non-zero determinant is denoted by

$$\mathbf{A}^{-1}$$

and satisfies

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}, \text{ and } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

The inverse of $\mathbf{A}$ is calculated as

$$\mathbf{A}^{-1} = |\mathbf{A}|^{-1}\mathbf{M}'_A$$

where $\mathbf{M}_A$ is the matrix of determinants of signed minors of $\mathbf{A}$.

Example Inversion

Let

$$\mathbf{A} = \begin{pmatrix} 6 & -1 & 2 \\ 3 & 4 & -5 \\ 1 & 0 & -2 \end{pmatrix}$$

Then

$$|\,\mathbf{A}\,| = -57$$

The determinants of signed minors are

$$\mathbf{M}_A = \begin{pmatrix} -8 & 1 & -4 \\ -2 & -14 & -1 \\ -3 & 36 & 27 \end{pmatrix}$$

and

$$\mathbf{A}^{-1} = \frac{1}{-57} \begin{pmatrix} -8 & -2 & -3 \\ 1 & -14 & 36 \\ -4 & -1 & 27 \end{pmatrix}$$

Determinants of Products

Two square matrices, **A** and **B**, with the same dimensions, then

$$|\,\mathbf{AB}\,| = |\,\mathbf{A}\,| \cdot |\,\mathbf{B}\,|$$

If $|\,\mathbf{A}\,| = 0$ or $|\,\mathbf{B}\,| = 0$, then

$$|\,\mathbf{AB}\,| = 0$$

If $|\,\mathbf{AB}\,| \neq 0$, then

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Linear Dependence

Rank

Rank of any matrix is the number of linearly independent rows and columns, a scalar number.

Rules

1. If the determinant of a square matrix is NOT zero, then rank is equal to the order of the matrix, full rank, and the inverse of the matrix exists.
2. If the determinant of a square matrix IS zero, or if the matrix is not square, then rank is not full, and an inverse of the matrix does NOT exist.
3. If a matrix has r rows and c columns and $r < c$, then the rank of the matrix can not be greater than r.

Examples of Dependence

- $$\mathbf{B} = \begin{pmatrix} 3 & 9 \\ 1 & 3 \end{pmatrix}$$

  Note that column 2 equals 3 times column 1, or that row 1 equals 3 times row 2. The determinant of **B** is 0.

- $$\mathbf{C} = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -6 & -2 \\ -3 & 7 & 4 \end{pmatrix}$$

  Note that column 3 equals the sum of column 1 and column 2 - a linear dependency. The determinant of **C** is 0. The rank of **C** can be 0, 1, or 2, but not 3, and not greater than 3.

- Not all dependencies can be spotted easily by visual observation.

Example Rank

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 & 2 \\ 6 & 9 & 5 & 1 \\ 8 & 7 & 0 & 4 \\ 3 & 1 & 0 & 5 \end{pmatrix}$$

which has a 0 determinant. Therefore, the rank is less than 4.
Must use elementary operators to reduce matrix to a triangular matrix using **Elementary Operator** matrices. The number of non-zero diagonal elements in the reduced matrix gives the rank of the matrix.

Elementary Operators

Elementary operator matrices are identity matrices that have been modified by one of three methods.

1. Let $\mathbf{E}_{11}(.25)$ be an elementary operator where the first diagonal element of an identity matrix has been changed to .25.
2. Let $\mathbf{E}_{ij}$ be an elementary operator where rows i and j are interchanged.
3. Let $\mathbf{E}_{ij}(c)$ be an elementary operator where c occupies row i and column j.

Multiplying a matrix by an elementary operator matrix does not change the RANK of the matrix.

Calculation of Rank, Example

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 & 2 \\ 6 & 9 & 5 & 1 \\ 8 & 7 & 0 & 4 \\ 3 & 1 & 0 & 5 \end{pmatrix}$$

The goal is to reduce **A** to an upper triangular form through successive pre-multiplications by elementary operator matrices. The first one is

$$\mathbf{P}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -6 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P}_1\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 & 2 \\ 0 & -9 & -25 & -11 \\ 8 & 7 & 0 & 4 \\ 3 & 1 & 0 & 5 \end{pmatrix}$$

The same type of elementary operators can be used to eliminate the 8 in row 3 and the 3 in row 4.

Calculation of Rank

The reduced form after premultiplication by 3 elementary operators is

$$\mathbf{P}_3\mathbf{P}_2\mathbf{P}_1\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 & 2 \\ 0 & -9 & -25 & -11 \\ 0 & -17 & -40 & -12 \\ 0 & -8 & -15 & -1 \end{pmatrix}$$

Now two more elementary operators will make -17 in row 3 and -8 in row 4 change to 0. Finally the sixth elementary operator will change the element in the 4^{th} row and 3^{rd} column equal to 0. The final reduced matrix is

$$\mathbf{P}_6\mathbf{P}_5\mathbf{P}_4\mathbf{P}_3\mathbf{P}_2\mathbf{P}_1\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 & 2 \\ 0 & -9 & -25 & -11 \\ 0 & 0 & \frac{65}{9} & \frac{79}{9} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The number of non-zero diagonal elements in this matrix is 3. Therefore, the rank of $\mathbf{A}$ is 3, or $r(\mathbf{A}) = 3$.

More About Rank

1. $\mathbf{A}_{10\times 50}$, then $r(\mathbf{A}) \leq 10$
2. $r(\mathbf{AA}') = r(\mathbf{A})$
3. $\mathbf{A}_{6\times 3}\mathbf{B}_{3\times 10} = \mathbf{C}_{6\times 10}$, then $r(\mathbf{C}) \leq 3$
4. $r(\mathbf{J}) = 1$
5. $r(\mathbf{0}) = 0$

Definitions

Full-row rank
: If $\mathbf{A}$ has order $m \times n$ with rank equal to m, then $\mathbf{A}$ has full row rank.

Full-column rank
: A matrix with rank equal to the number of columns has full-column rank.

Full rank
: A square matrix with rank equal to the number of rows or columns has full rank. A full rank matrix is nonsingular, has a non-zero determinant, and has an inverse.

Partitioning A Matrix

$\mathbf{A}$ of order $p \times q$, rank r, and r is less than or equal to the smaller of p or q.

$$\mathbf{A}_{p\times q} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

$\mathbf{A}_{11}$ order $r \times r$ and rank r. Re-arrangement of rows and columns of $\mathbf{A}$ may be needed to find an appropriate $\mathbf{A}_{11}$.
$\mathbf{A}_{12}$ order $r \times (q - r)$
$\mathbf{A}_{21}$ order $(p - r) \times r$, and
$\mathbf{A}_{22}$ order $(p - r) \times (q - r)$

Dependencies in Partitions

$$\begin{pmatrix} \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} = \mathbf{K}_2 \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A}_{12} \\ \mathbf{A}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11} \\ \mathbf{A}_{21} \end{pmatrix} \mathbf{K}_1$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{11}\mathbf{K}_1 \\ \mathbf{K}_2\mathbf{A}_{11} & \mathbf{K}_2\mathbf{A}_{11}\mathbf{K}_1 \end{pmatrix}$$

Generalized Inverses

$$\mathbf{A}\mathbf{A}^{-}\mathbf{A} = \mathbf{A}$$

Moore-Penrose inverse satisfies:

1. $\mathbf{A}\mathbf{A}^{-}\mathbf{A} = \mathbf{A}$,
2. $\mathbf{A}^{-}\mathbf{A}\mathbf{A}^{-} = \mathbf{A}^{-}$,
3. $(\mathbf{A}^{-}\mathbf{A})' = \mathbf{A}^{-}\mathbf{A}$, and
4. $(\mathbf{A}\mathbf{A}^{-})' = \mathbf{A}\mathbf{A}^{-}$.

Compute the generalized inverse as

$$\mathbf{A}^{-} = \begin{pmatrix} \mathbf{A}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Generalized Inverses

If the generalized inverse satisfies only one of the Moore-Penrose conditions, then there are an infinite number of generalized inverses for any non full rank matrix.

If $\mathbf{G}$ represents the generalized inverse, then other generalized inverses, $\mathbf{F}$, can be obtained from

$$\mathbf{F} = \mathbf{GAG} + (\mathbf{I} - \mathbf{GA})\mathbf{X} + \mathbf{Y}(\mathbf{I} - \mathbf{AG})$$

for any arbitrary matrices $\mathbf{X}$ and $\mathbf{Y}$ of the correct order.

Other Things in Notes

- Eigenvalues and eigenvectors
- Differentiation, tomorrow
- Cholesky decomposition